[4] Friedrichs, K. O. and Stoker, J. J., "The non-linear boundary value problem of the buckled plate," Amer. J. Math. 63, 839–888 (1941).

[5] Wagner, N. R., "Existence proof for a nonlinear boundary value problem in ordinary differential equations," Ph.D. dissertation Mass. Inst. Tech. (1961).

[6] Yanowitch, M., "Non-linear buckling of circular elastic plates," Communs. Pure and Appl. Math. 9, 661–672 (1956).

Low-Frequency Acoustic Oscillatory Combustion

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A description is made of a large, double-end burner that is used for the study of low-frequency combustion instability. Also described are the techniques applied and some results obtained by the use of the system.

Introduction

OSCILLATORY combustion is a problem that has frequently been encountered in the development of solid propellant rockets. Accordingly, techniques were developed (for example, Refs. 1–4) whereby oscillatory combustion was studied on a laboratory scale in self-excited systems. Subsequently, techniques have been developed whereby externally excited systems are used to study the more stable propellants.[5,6]

In general, the foregoing techniques have been suitable for studying either the higher acoustic (above 500 cps) frequencies or nonacoustic frequencies. Because of advances in the state of the art, larger motors are now being built. Although a great deal has been learned about acoustic oscillatory combustion in the smaller motors (which exhibit higher frequencies), there is virtually nothing known about acoustic oscillatory combustion at the low frequencies that are of concern in large motors. There has been no laboratory test that would reveal the information, nor has it been gained from motor firings, as they have not yet been made.

Knowledge gained through the trial-and-error methods of development programs is not only extremely costly and time consuming, but it is also generally inadequate for research purposes, and, accordingly, new techniques have been developed.[7,8] With these techniques, it is possible to study acoustic oscillatory combustion in the 5- to 120-cps range with all propellants. This paper is devoted to a discussion of the test burner, techniques, and results obtained.

Apparatus

The burner employed (see Fig. 1) is a $5\frac{1}{2}$-in.-i.d. cylindrical steel tube that is segmented so that it may be used in lengths of 12, 24, 36, 48, or 60 ft. Propellant grains $5\frac{3}{8}$-in. o.d. and usually 1-in. thick (about 2 lb) are used in the burner. The grain is cemented into the end of the burner with epoxy resin. In a given test a single grain may be used in one end of the burner, or grains may be placed in both ends of the burner. Ignition is accomplished by means of an electrically heated bridge-wire that ignites a pyrotechnic paste spread on the propellant surface and coated on the wire.

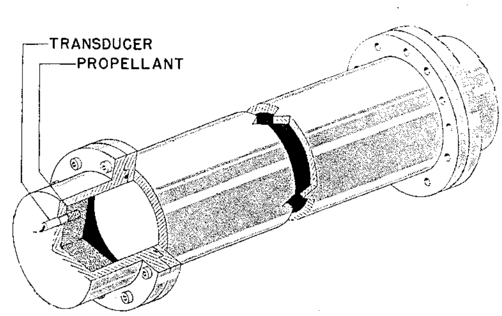

Fig. 1 Test burner used to evaluate a propellant for low-frequency, oscillatory combustion.

The rather simple instrumentation consists of two channels. The main channel is a low-frequency-response pressure transducer whose output signal is recorded on a galvanometer oscillograph. Also recorded on the oscillograph is the amplified and filtered output signal from a high-frequency-response pressure transducer. This latter channel largely performs a "back-up" function for the former channel. Information concerning the pressure-time history (including the pressure oscillations) is taken from the oscillograph record.

Techniques

Depending upon the information desired from the test and the nature of the propellant, there are three types of experimental techniques that are employed. These involve tests that probe the pressure-frequency spectrum for areas of self-excited instability, tests that yield the types of quantitative data which were previously obtained only at higher frequencies,[9] and tests of propellants that are stable in the system so that the propellants may be rated as to their degree of stability.

The exploratory type of test is performed by firing the test in an unvented burner. Following ignition, the closed system increases in pressure until the propellant is consumed. The test will show if, over the pressure range tested and at the characteristic frequency of the burner, the propellant combustion will generate acoustic pressure oscillations in the system. When it is desired that the rate of pressurization in the burner be small, a small bleed or "pseudo-nozzle" is placed midway between the burner ends. Figure 2 shows a record from a test of this nature. By the use of the various burner lengths, the low-frequency end of the spectrum can be probed systematically.

Using the burner, one can also extend to lower frequencies the data that characterize the degree of instability of the propellant. In order to obtain these data, it is necessary that the system be brought to pressure rapidly after ignition and that the oscillations be allowed to grow spontaneously. Because of large free volume and high heat transfer, the sys-

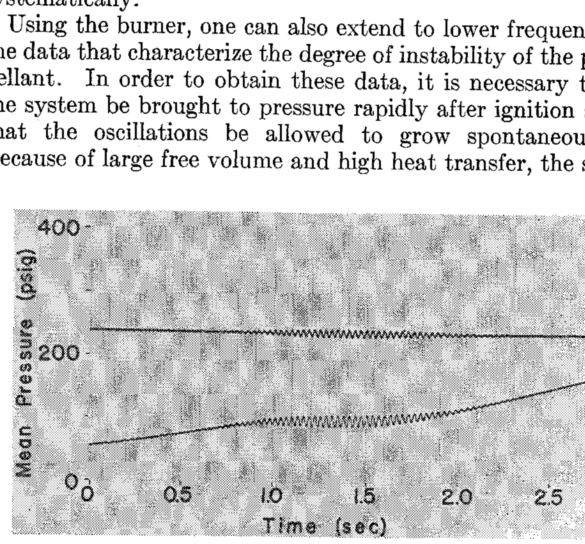

Fig. 2 Portion of a test record obtained from a firing in the sealed burner. The propellant employed showed a marked selectivity in the conditions under which it would oscillate.

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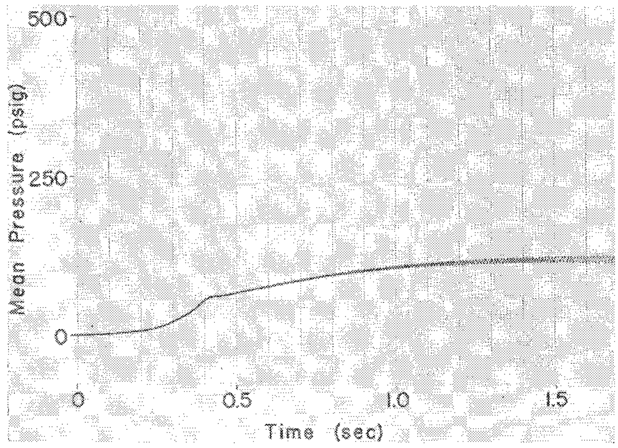

Fig. 3 Portion of a test record which illustrates rapid pressurization of the burner through the use of propellant shavings.

tem will not come quickly to equilibrium if ignited in the normal fashion. Therefore, to bring the system rapidly to pressure, small amounts of the propellant shavings are placed near each propellant surface. Upon ignition, both the propellant disks and shavings begin to burn. The shavings are consumed within a very short interval of time and cause the mean pressure of the system to rise to the desired operating level. Thereafter, constant pressure is maintained by a properly sized nozzle placed midway between the two ends of the burner.

The pressure oscillations in the burner (see Fig. 3) then grow exponentially to high amplitude as

$$p_m = p_0 \exp\alpha_g t$$

where p_m is the amplitude of the acoustic pressure, p_0 is the amplitude at an arbitrary zero time, t is time, and α_g is the growth rate constant. The growth rate constant is a measure of the acoustic energy accumulation in the system and can be determined from a test record.[9] Similarly, a decay constant that characterizes the losses in the system may be evaluated shortly after the propellant is consumed. Combination of the growth and decay constants will permit the analytical classification of the instability of the propellant combustion.[9]

The burner may also be used to classify the relative degree of stability of propellants that are stable in the system. Although this burner is, compared to a rocket motor, an acoustically conservative system, it is also much more weakly driven, since the driving is proportional to the propellant area in a system. Accordingly, there is no assurance that a propellant that is stable in this burner would also be stable in a rocket motor, and hence it is very desirable that the system be capable of testing "stable" propellants. In order to make such tests, the system is brought to pressure, given a finite

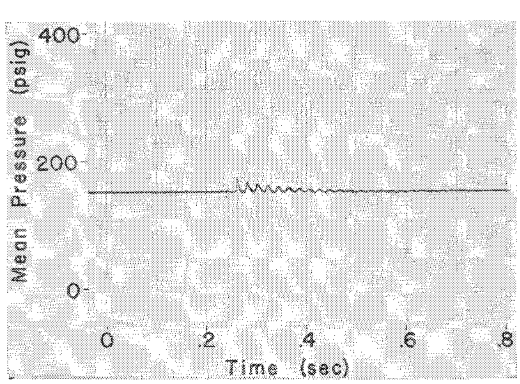

Fig. 4 Portion of a test record which shows the oscillations produced by the use of a black powder charge.

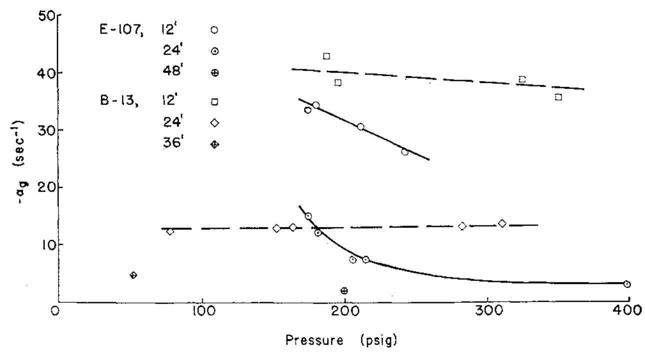

Fig. 5 Growth constants determined by the pulse technique under various test conditions; E-107 is a highly aluminized, polyurethane-ammonium perchlorate composite propellant, and B-13 is a highly aluminized, ammonium perchlorate-polybutyl acrylic acid composite propellant.

pressure pulse, and the rate of decay of the pulse is measured. To accomplish this, the burner nozzle is sealed by a device capable of release upon demand. The system is then prepressurized with nitrogen to a carefully chosen level. Upon ignition, the nozzle seal is released, and after the nitrogen has been flushed from the burner by the combustion gases, the system is approximately at its equilibrium pressure. Shortly thereafter a charge of black powder is ignited at one end of the burner (see Ref. 5 for technique details), and the resultant gases generate a pressure pulse that after about two "cycles" becomes shaped into a first-mode, longitudinal acoustic oscillation. Subsequently, the decaying oscillations are analyzed and a growth constant α_g determined. Note that, inasmuch as the oscillations are decaying, α_g is negative.

Figure 4 shows the pressure-time history of such a pulse, and Fig. 5 shows α_g values for two propellants at several pressures and frequencies. This latter figure shows that the system is indeed capable of revealing differences between the behavior of two propellants and that it is the characteristics of the propellant, rather than of the burner, which largely determine the composite behavior of the system. Also, Fig. 5 shows that use of the techniques will reveal details concerning the dynamic combustion characteristics of a given propellant. If the E-107 data are interpolated at 200 psig and replotted as shown in Fig. 6, the data suggest that in burners longer than about 55 ft the propellant will spontaneously generate oscillations. Such was found to be the case when the propellant was tested in the 60-ft burner (12 cps).

Conclusion

The burner and techniques described herein offer considerable promise for the study of low-frequency acoustic combustion instability. Furthermore, they provide a means

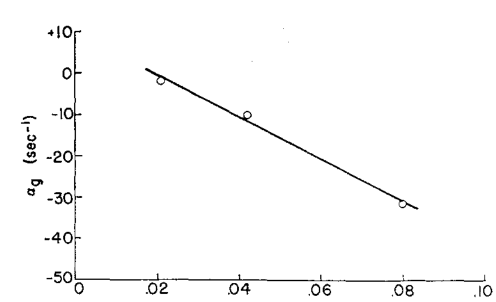

Fig. 6 Cross plot of E-107 data from Fig. 4 for 200 psig. Extrapolation predicts the observed tendency of the propellant to generate self-excited oscillations in the 60-ft burner.

whereby a few pounds of any propellant may be used to investigate oscillatory combustion in the frequency range of large, solid propellant boosters designed to employ several tons of propellant. Such a tool then may materially aid the development of large systems by providing propellant evaluation without expensive full-scale motor firings.

References

1 Price, E. W. and Sofferis, J. E., "Combustion instability in solid propellant rocket motors," Jet Propulsion **28**, 190–193 (1958).

2 Angelus, T. A., "Panel discussion on solid propellant combustion instability," *Eighth Symposium (International) on Combustion* (Williams and Wilkins Co., Baltimore, Md., 1962), pp. 921–924.

3 Wood, W. A., "Panel discussion on solid propellant combustion instability," *Eighth Symposium (International) on Combustion* (Williams and Wilkins Co., Baltimore, Md., 1962), pp. 918–920.

4 Ryan, N. W., "Panel discussion on solid propellant combustion instability," *Eighth Symposium (International) on Combustion* (Williams and Wilkins Co., Baltimore, Md., 1962), pp. 924–925.

5 Dickinsen, L. A. and Jackson, F., "Combustion in solid propellant rocket engines," *Fifth AGARD Combustion and Propulsion Colloquium* (Pergamon Press, London, 1962), pp. E1–E14.

6 Watermeier, L., "Panel discussion on solid propellant combustion instability," *Eighth Symposium (International) on Combustion* (Williams and Wilkins Co., Baltimore, Md., 1962), pp. 920–921.

7 Price, E. W., "NOTS Research on low frequency combustion instability," Second Meeting of Technical Panel on Solid Propellant Combustion Instability, Stanford Res. Inst., Menlo Park, Calif., pp. 43–50 (March 8–9, 1962).

8 Price, E. W., "Low frequency combustion instability of solid rocket propellants," U. S. Naval Ordnance Test Station, China Lake, Calif., TR 301 (December 1962).

9 Horton, M. D. and Price, E. W., "Dynamic characteristics of solid propellant combustion," *Ninth Symposium (International) on Combustion* (Academic Press, New York, 1963), pp. 303–310.

Deformations and Stresses in Axially Loaded and Heated Cylindrical Shells

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The authors have recently conducted some numerical studies to assess the effects of axial thrust on the bending stresses and deflections in cylindrical booster tanks subjected to longitudinal temperature variations. The calculations were based on the following displacement equation of equilibrium:

$$D\frac{d^4w}{dx^2} - N_x\frac{d^2w}{dx^2} + \frac{Ehw}{R^2} = \frac{\nu N_x - N_t}{R} \quad (1)$$

where

$N_t = \int_{-h/2}^{h/2} \alpha ET(x)dz$

x = axial coordinate measured from a cylinder edge

z = thickness coordinate, positive inward

h,R = thickness and radius of cylinder, respectively

and the remaining notation is the same as in Ref. 1. The second term on the left side of Eq. (1) represents the additional transverse loading due to N_x which is included in the classical buckling equation.

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With the assumption that the edges be free of restraint, the boundary conditions are given by

$$\left.\begin{array}{l} d^2w/dx^2 = 0 \\ (d^3w/dx^3) - (N_x/D)(dw/dx) = 0 \end{array}\right\} \text{at } x = 0, x = L \quad (2)$$

Assume that N_t is analytic and thereby expressible in the form of a power series. The nondimensional solution of (1) is

$$\bar{w} = C_1 \cosh\gamma\xi \cos\delta\xi + C_2 \sinh\gamma\xi \cos\delta\xi + C_3 \cosh\gamma\xi \sin\delta\xi + C_4 \sinh\gamma\xi \sin\delta\xi + \sum_{k=0}^{\infty}\frac{B_k}{\lambda^2}\sum_{j=0,1,\ldots}^{[k/2]} A_{k-2j}\,\xi^{k-2j} \quad (3)$$

where

$$\bar{N}_t = -\sum_{k=0}^{\infty} B_k\xi^k = \frac{12(1-\nu^2)}{ER}\left(\frac{L}{h}\right)^4 N_t$$

$$\xi = \frac{x}{L}$$

$$\lambda^2 = 12(1-\nu^2)\left(\frac{L}{R}\right)^4\left(\frac{R}{h}\right)^2$$

$$\beta = \frac{-N_xR[3(1-\nu^2)]^{1/2}}{Eh^2}$$

$$w = \frac{w}{h} - \frac{\nu R}{Eh^2}N_x$$

$$\gamma = \left[\frac{\lambda}{2}(1-\beta)\right]^{1/2} \qquad \delta = \left[\frac{\lambda}{2}(1+\beta)\right]^{1/2}$$

and C_i are arbitrary constants. The quantities A_{k-2j} are given by[2]

$$A_{k-2j} = \frac{(-1)^j k!}{(k-2j)!}\left[\left(\frac{2\beta}{\lambda}\right)^j + \sum_{r=2}^{[(j/2)+1]}\frac{(-1)^{r-1}(j-r+1)(j-r)\ldots(j+3-2r)}{(r-1)!\,4^{r-1}} \times \left(\frac{2\beta}{\lambda}\right)^{j-2(r-1)}\left(\frac{4}{\lambda^2}\right)^{r-1}\right]$$

$[N]$ is the largest integer less than or equal to N.

The dimensionless bending stresses in the longitudinal and hoop directions are calculated from

$$\bar{\sigma}_{xb} = -\left(\frac{3}{1-\nu^2}\right)^{1/2}\frac{1}{\lambda}\frac{d^2\bar{w}}{d\xi^2} = \frac{\bar{\sigma}_{\phi b}}{\nu} \quad (4)$$

where

$$(\bar{\sigma}_{xb},\bar{\sigma}_{\phi b}) = (R/Eh)(\sigma_{xb},\sigma_{\phi b})$$

These must be superposed on the direct stresses

$$\bar{\sigma}_{xd} = -\frac{\beta}{[3(1-\nu^2)]^{1/2}} \qquad \bar{\sigma}_{\phi d} = -\left(\bar{w} + \frac{N_t}{\lambda^2}\right) \quad (5)$$

where

$$(\bar{\sigma}_{xd},\bar{\sigma}_{\phi d}) = (R/Eh)(\sigma_{xd},\sigma_{\phi d})$$

Deflections for a temperature of the form $T = T_0\xi^2$ are shown in Fig. 1. Axially loaded to unloaded bending stress ratios for this temperature distribution and, in addition, for $T = T_0\xi^3$ are given in Fig. 2. All results are for a cylinder geometry parameter of $\lambda = 600$ (corresponding, for example, to $R/h \approx 1000$ and $L/R \approx 0.40$). It is seen that the deflections are rather insensitive to end load, whereas the bending stresses due to end load may be many times that predicted by elementary superposition. For example, if $T/T_0 = \xi^2$, this ratio is 7.5 for $\beta = 0.2$ (0.4 of the buckling load),[3] showing the adverse effect of the end loads.